Finitary functors: from Set to Preord and Poset

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Motivation

- Most of coalgebraic logic is focussed on Set-coalgebras and their associated (Boolean) logics.
- Investigation of coalgebraic logic over Poset already started – expressivity results [Kurz-Kapulkin-Velebil CMCS2010].
- Would deserve a systematic investigation of Poset-functors and their coalgebras.
- In this talk: we restrict on how to move from (finitary) Set-functors (fairly-well understood) to Preord and Poset-functors with a quick look on their properties and coalgebras.

Outline

1. Extensions and liftings
2. From Set-functors to Preord-functors
   - Order on variables
   - Order on operations
   - Order both variables and operations
3. Finally, from Preord to Poset
4. Further work

Extensions and liftings

- We fix a *Set*-functor T
- Recall the adjunction $\begin{tikzcd} \mathrm{Set} \arrow[r, shift left, "D"] \arrow[r, phantom, "\perp"] & \mathrm{Preord} \arrow[l, shift left, "U"] \end{tikzcd}$

Extension:
$$\begin{tikzcd} \mathrm{Preord} \arrow[r, "\Gamma"] & \mathrm{Preord} \\ \mathrm{Set} \arrow[u, "D"] \arrow[r, "T"] & \mathrm{Set} \arrow[u, "D"] \end{tikzcd}$$

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- If Γ : Preord $\to$ Preord is a lifting/extension of T, then $T = U\Gamma D$.

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- What about the composition $\Gamma = DTU$?

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- We require for Γ (lifting or extension) to be locally monotone and also finitary if T is finitary.
- If $\Gamma : \mathrm{Preord} \to \mathrm{Preord}$ is a lifting/extension of T, then $T = U\Gamma D$.
- What about the composition $\Gamma = DTU$? DTU is not locally monotone.

Example of a finitary Set-functor having an extension which is not finitary

- Consider the functor $T : \mathsf{Set} \to \mathsf{Set}$,

$$TX = \{l : \mathbb{N} \to X \mid l(n) = l(n+1) \text{ for all but a finite number of } n\}$$

- T is finitary

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- T is finitary and has the Preord-extension

$$\Gamma(X, \leq) = \{l : (\mathbb{N}, \leq) \to (X, \leq) \mid l(n) \leq l(n+1) \\ \text{for all but a finite number of } n\}$$

  with the pointwise order.

- But Γ is not finitary: take the sequence

$$(\underline{1}, \leq) \subseteq (\underline{2}, \leq) \subseteq \ldots \longrightarrow (\mathbb{N}, \leq)$$

- Then $\Gamma(\mathbb{N}, \leq) \ncong \mathrm{colim}\Gamma(\underline{n}, \leq)$.

More on extensions/liftings

Extensions and liftings are not unique.

Examples:

Extension

$T = \mathrm{Id}$

$\Gamma_1 = \mathrm{Id}$

Γ_2=(discrete) connected component functor

Lifting

$TX = 2 \times X$

$\Gamma_1(X, \leq) = \mathbf{2} \times X$, product order

$\Gamma_2(X, \leq) = \mathbf{2} \ltimes X$, lexicographic order

About coalgebras

Γ extension of T

$$\mathrm{Set} \underset{C}{\overset{D}{\rightleftarrows}} \mathrm{Preord} \quad (D \dashv C)$$

$$\mathrm{Coalg}(T) \underset{\tilde{C}}{\overset{\tilde{D}}{\rightleftarrows}} \mathrm{Coalg}(\Gamma) \quad (\tilde{D} \dashv \tilde{C})$$

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$$\mathrm{Coalg}(T) \underset{\tilde{U}}{\overset{\tilde{D}}{\rightleftarrows}} \mathrm{Coalg}(\Gamma) \quad (\tilde{D} \vdash \tilde{U})$$

Final Γ-coalgebra is the final T-coalgebra with some preorder.

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First construction: order on variables

T finitary Set-functor $\iff$ quotient of a polynomial functor.

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$$\coprod_{m,n<\omega} \mathrm{Set}(\underline{m},\underline{n}) \times T\underline{m} \times X^n \rightrightarrows \coprod_{n<\omega} T\underline{n} \times X^n \longrightarrow TX$$

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Obtain functor $\tilde{T}(X, \leq) = (TX, \trianglelefteq) : \mathsf{Preord} \to \mathsf{Preord}$

- Locally monotone
- Both lifting and extension
- Call $\tilde{T}$ the *preordification* of T

Proposition

$\tilde{T}$ *is independent of the chosen presentation of* T.

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Examples

- $TX = X^*$
  Then $\trianglelefteq$ compares lists of same length element by element:

$$[x_0 \dots x_{n-1}] \trianglelefteq [y_0 \dots y_{m-1}] \;\Leftrightarrow\; m = n \,\wedge\, x_i \le y_i,\; \forall i < n$$

- $TX = \mathcal{P}_f X$
  Then $\trianglelefteq$ is the Egli-Milner preorder on $\mathcal{P}_f(X, \le)$:

$$u \trianglelefteq v \text{ for } u, v \subseteq X \text{ finite } \Leftrightarrow \begin{cases} \forall a \in u\; \exists b \in v. & a \le b \\ \forall b \in v\; \exists a \in u. & a \le b \end{cases}$$

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T finitary Set functor.

- Take $(T\underline{n}, \leq)$ preordered such that $Tf : (T\underline{m}, \leq) \to (T\underline{n}, \leq)$ is monotone for any map $f : \underline{m} \to \underline{n}$.

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- Motivation: there are natural examples, like $\mathcal{P}_f$ with inclusion.
- But also easy general example:

$$
\begin{tikzcd}
T\underline{1} \neq \emptyset \arrow[r] & \begin{array}{c}\text{pick } \leq \\ \text{preorder on } T\underline{1}\end{array} \arrow[r] & \begin{array}{c}\text{obtain } \leq \\ \text{preorder on } T\underline{n}\end{array} \\
\underline{n} \xrightarrow{!} \underline{1} \arrow[r] & T\underline{n} \xrightarrow{T!} T\underline{1} \arrow[ur] &
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Order:

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  Not yet a lifting!

Proposition

- *The preorder on the signature is recovered.*
- *There is an one-to-one correspondence*
  order $\bar{T}X = (TX, \sqsubseteq) \Longleftrightarrow$ *preorder on signature* $(T\underline{n}, \leq)_{n<\omega}$

Second ingredient in second construction: T–relators

- For relation $R \subseteq X \times Y$, the T-relation lifting $\mathrm{Rel}_T(R) \subseteq TX \times TY$ is described as

$$(u, v) \in \mathrm{Rel}_T(R) \Leftrightarrow \exists w \in TR.\ T\pi_1(w) = u \ \wedge \ T\pi_2(w) = v,$$

where $X \xleftarrow{\pi_1} R \xrightarrow{\pi_2} Y$.

- Now: assume order $\bar{T}X = (TX, \sqsubseteq)$ on T.
- For any relation $R \subseteq X \times Y$, the T-relator $\mathrm{Rel}_T^{\sqsubseteq}(R) \subseteq TX \times TY$ is given by

$$(u, v) \in \mathrm{Rel}_T^{\sqsubseteq}(R) \ \Leftrightarrow \ \exists w \in T(R).\ u \sqsubseteq T\pi_1(w) \ \wedge \ T\pi_2(w) \sqsubseteq v$$

[Thijs 1996, Hughes-Jacobs 2004]

Properties of T-relators

- $\sqsubseteq_{TX} \ = \ \mathrm{Rel}^{\sqsubseteq}_{T}(=_X)$. If $R \subseteq S$ then $\mathrm{Rel}^{\sqsubseteq}_{T}(R) \subseteq \mathrm{Rel}^{\sqsubseteq}_{T}(S)$.

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- For any functions $f : X \to X'$, $g : Y \to Y'$ and any relation $R' \subseteq X' \times Y'$, $\mathrm{Rel}^{\sqsubseteq}_{T}((f \times g)^{-1}(R')) \subseteq (Tf \times Tg)^{-1}(\mathrm{Rel}^{\sqsubseteq}_{T}(R'))$

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- If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$, then $\mathrm{Rel}_T^{\sqsubseteq}(S \circ R) \subseteq \mathrm{Rel}_T^{\sqsubseteq}(S) \circ \mathrm{Rel}_T^{\sqsubseteq}(R)$

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  If holds with equality, say that the order $\bar{T}$ *preserves composition of preorders*.

And now comes the lifting...

- Let $\bar{T}X = (TX, \sqsubseteq)$ an order on T.
- Assume $\bar{T}$ preserves composition of preorders.
- Obtain Preord-lifting of T given by $\hat{T}(X, \leq) = (TX, \mathrm{Rel}^{\sqsubseteq}_{T}(\leq))$.

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Examples

- If the order on T is discrete and T preserves weak pullbacks, then $\mathrm{Rel}_T(\leq) = \trianglelefteq$ and consequently $\hat{T} = \tilde{T}$ Preordification

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Examples

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- If the order is indiscrete then $\hat{T}(X, \leq) = (TX, TX \times TX)$

More examples

- $T = \mathcal{P}_f$
  Order: the inclusion; stable.
  Lifting with: $(u, v) \in \mathrm{Rel}^{\subseteq}_{\mathcal{P}_f}(\leq) \;\Leftrightarrow\; \forall a \in u \; \exists b \in v \; . \; a \leq b$, where $u, v \in \mathcal{P}_f X$ and $(X, \leq)$ is preordered.

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- $TX = \mathbb{N} \times X$
  Order: lexicographic; not stable, but preserves composition of preorders.
  Lifting: $\hat{T}(X, \leq) = \mathbb{N} \ltimes X$ lexicographically ordered.

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- $TX = \mathbb{N} \times X$
  Order: lexicographic; not stable, but preserves composition of preorders.
  Lifting: $\hat{T}(X, \leq) = \mathbb{N} \ltimes X$ lexicographically ordered.
- $T = (-)^3_2$
  Order: zig-zag

$$
\begin{tikzcd}
\cdots \arrow[dr, dash] & & (x_1, x_2, x_1) \arrow[dl, dash] \arrow[dr, dash] & & (x_2, x_2, x_3) \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dl, dash] \\
& (x_1, x_1, x_1) & & (x_2, x_2, x_2) & & (x_3, x_3, x_3) &
\end{tikzcd}
$$

  Order does not preserve composition of preorders, thus $\mathrm{Rel}^{\sqsubseteq}_T(\leq)$ is not necessarily a preorder, for $(X, \leq)$ preordered set.
  No lifting using relators.

Preservation-type properties of lifted functor

Set: weak pullback-preserving functors

Weak pullback:
$$\begin{array}{ccc} P & \xrightarrow{\alpha} & X \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$
with $f\alpha = g\beta$, such that

$\forall x \in X, y \in Y.\ f(x) = g(y) \Rightarrow \exists p \in P.\ x = \alpha(p) \wedge \beta(p) = y$

Preservation-type properties of lifted functor

Preord: exact square-preserving functors

Exact square:

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\beta"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

with $f\alpha \leq g\beta$, such that

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Preservation-type properties of lifted functor

Preord: exact square-preserving functors

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Proposition

Let T be a finitary Set-functor having an order $\bar{T}(X, \leq) = (TX, \sqsubseteq)$ which preserves composition of preorders. Then the following are equivalent:

1. *The order is stable.*
2. *The order maps weak pullbacks to exact squares.*
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Consequence: if T is a finitary Set-functor, then T preserves weak pullbacks if and only if its preordification $\tilde{T}$ preserves exact squares.

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Let T be a Set*-functor (not necessarily finitary). There is a bijective correspondence between:*

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Corollary

Order on T is stable $\Longrightarrow$ $\hat{T}$ preserves embeddings.

Remark: Converse is false: take for example

$$TX = \{*\} \ + \ (X \times X - \Delta_X)/_\sim$$

Then T fails to preserve weak pullbacks, thus the discrete order on T is not stable, but $\tilde{T}$ does preserve embeddings (notice that $\tilde{T}(X, \leq)$ is ordered component-wise with $*$ as bottom element).

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- Recall: for T finitary Set-functor and Γ a lifting of T to Preord, the final Γ-coalgebra exists and has the final T-coalgebra as underlying set.

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  - Recall: given an order $\bar{T}X = (TX, \sqsubseteq)$ and two T-coalgebras $X \xrightarrow{c} TX$, $Y \xrightarrow{d} TY$, a relation $R \subseteq X \times Y$ is called a T-simulation wrt order $\bar{T}$ iff

$$\begin{tikzcd} x \arrow[d, squiggly] & y \arrow[d, squiggly] \\ c(x) & d(y) \end{tikzcd}$$

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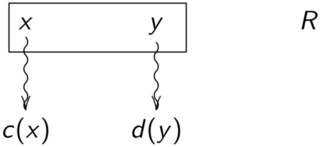

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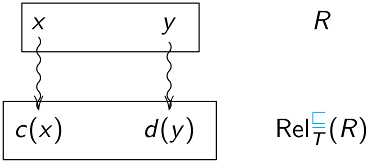

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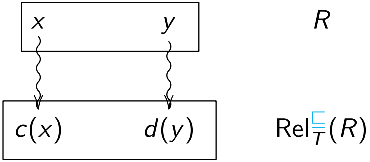

  - Similarity: greatest simulation. Similarity on a T-coalgebra $X \to TX$ is a preorder.

Outline

1. Extensions and liftings
2. From Set-functors to Preord-functors
   - Order on variables
   - Order on operations
   - Order both variables and operations
3. Finally, from Preord to Poset
4. Further work

Third construction: order both variables and operations

T finitary Set-functor

$$\coprod_{m,n<\omega} \mathrm{Set}(\underline{m},\underline{n}) \times T\underline{m} \times X^n \rightrightarrows \coprod_{n<\omega} T\underline{n} \times X^n \to TX$$

Third construction: order both variables and operations

T finitary Set-functor $\quad (X, \leq)$ preordered set

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Preorder on signature $(T\underline{n}, \sqsubseteq)_{n<\omega}$

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Obtain T-lifting $\check{T}(X, \leq) = (TX, \preceq)$

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Obtain T-lifting $\check{T}(X, \leq) = (TX, \preceq)$

Proposition

If T preserves weak pullbacks and $\bar{T}$ preserves composition of preorders, then $\check{T} = \hat{T}$.

▸ Relator Lifting

Remark: still a T-lifting independently of the properties of the order $\bar{T}$.

▸

Outline

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$$\begin{tikzcd}
\mathsf{Set} \arrow[loop above, "T"] & & \mathsf{Preord} \arrow[loop above, "\Gamma"] \arrow[r, shift left, "Q"] \arrow[r, phantom, "\perp"] & \mathsf{Poset} \arrow[l, shift left, "J"]
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Define $T' = Q\Gamma J$ locally monotone, finitary.

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$$\begin{tikzcd} \mathrm{Coalg}(T) \arrow[r, shift left] \arrow[r, phantom, "\top"] & \mathrm{Coalg}({\color{blue}\Gamma}) \arrow[l, shift left] \arrow[r] & \mathrm{Coalg}(T') \end{tikzcd}$$

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- Γ extension of T to Preord $\Rightarrow$ T' extension of T to Poset. Final T'-coalgebra exists and is discrete.
- Example: for $T = \mathcal{P}_f$ and $\Gamma = \tilde{P}_f$, we obtain that $\mathcal{P}'_f$ is the finitely generated convex powerset functor

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- Γ lifting of T, then T' is not necessarily a lifting, just a quotient.
- However, if we have an order preserving composition of preorders and satisfying $\mathrm{Rel}^{\sqsubseteq}_{T}(R_1) \cap \mathrm{Rel}^{\sqsubseteq^{op}}_{T}(R_2) \subseteq \mathrm{Rel}_T(R_1 \cap R_2)$ then the lifting $\hat{T}$ restricts to posets.

Further work

Investigate coalgebraic logic over Poset and merge it with Set-based functors' logic into a big picture.

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