Relation lifting on preorders, metric spaces, etc.

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joint work with
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The characterisation theorem (V. Trnková 1977)

For a functor $T : \mathsf{Set} \longrightarrow \mathsf{Set}$, the following are equivalent:

1. There is a functor $\overline{T} : \mathsf{Rel}(\mathsf{Set}) \longrightarrow \mathsf{Rel}(\mathsf{Set})$ such that the square

$$\begin{array}{ccc} \mathsf{Rel}(\mathsf{Set}) & \overset{\overline{T}}{\dashrightarrow} & \mathsf{Rel}(\mathsf{Set}) \\ {\scriptstyle (-)_\diamond}\uparrow & & \uparrow{\scriptstyle (-)_\diamond} \\ \mathsf{Set} & \underset{T}{\longrightarrow} & \mathsf{Set} \end{array}$$

   commutes.

2. T preserves weak pullbacks.

Here, for $f : A \longrightarrow B$, $f_\diamond(b, a) = 1$ iff $b = fa$.

Where is relation lifting useful?

The semantics of Moss' coalgebraic language with ∇, for $T : \mathsf{Set} \longrightarrow \mathsf{Set}$

1. The modal language $\mathcal{L}$

$$\varphi ::= p \mid \top \mid (\varphi \wedge \varphi) \mid (\neg\varphi) \mid \nabla\alpha$$

   for $p \in \mathsf{At}$, $\alpha \in T\mathcal{L}$.

2. Semantics in a coalgebra $c : X \longrightarrow TX$. Define

$$x \Vdash \nabla\alpha \quad \text{iff} \quad c(x)\ \overline{T}(\Vdash)\ \alpha$$

   for every $x \in X$, $\alpha \in T\mathcal{L}$.

Liftings of relations $\overline{T}(\in)$ and $\overline{T}(\leq)$ are used formulating proof systems for Moss' logics.

Where is relation lifting useful?

Characterizing bisimulation: B is a bisimulation between $c : X \to TX$ and $d : Y \to TY$ iff

$$B(x, y) \text{ implies } \overline{T}(B)(c(x), d(y)).$$

The largest bisimulation on $c : X \to TX$ is the largest fixed point of the operator

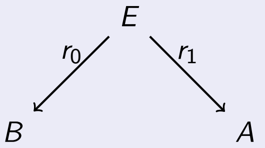
$$(c \times c)^{-1} \circ \overline{T}\ (-)$$

Definition

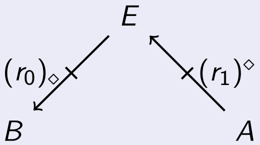
A relation from A to B is a map $R : B \times A \longrightarrow 2$, denoted by

$R : A \longrightarrow\!\!\!\!\!\!+\;\; B$

Relation R is tabulated by the span



if $R =$



where $(r_0)_\diamond(b, e) = 1$ iff $b = r_0(e)$, $(r_1)^\diamond(e, a) = 1$ iff $r_1(e) = a$.

Weak pullbacks

A square
$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{p_1} & \mathcal{B} \\ {\scriptstyle p_0}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathcal{A} & \xrightarrow[f]{} & \mathcal{C} \end{array}$$
in Set is a weak pullback

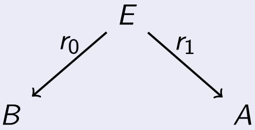
iff the square
$$\begin{array}{ccc} \mathcal{P} & \xleftarrow{(p_1)^{\diamond}} & \mathcal{B} \\ {\scriptstyle (p_0)_{\diamond}}\big\downarrow & & \big\downarrow{\scriptstyle (g)_{\diamond}} \\ \mathcal{A} & \xleftarrow[(f)^{\diamond}]{} & \mathcal{C} \end{array}$$
commutes in Rel(Set)

or, equivalently, iff for every a and b

$$fa = gb \text{ iff there exists } w \text{ s.t. } a = p_0(w) \text{ and } p_1(w) = b.$$

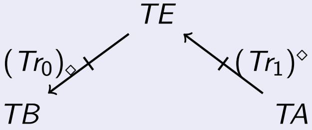
Definition of $\overline{T}$

Suppose $R : A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ is tabulated by



Define $\overline{T}(R) : TA \longrightarrow\!\!\!\!\!\!\!+\;\; TB$

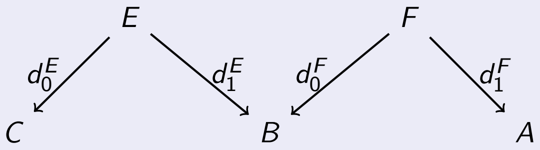
by putting



$$\overline{T}(R)(\beta, \alpha) = \bigvee_{w} (\beta = Tr_0(w)) \wedge (Tr_1(w) = \alpha)$$

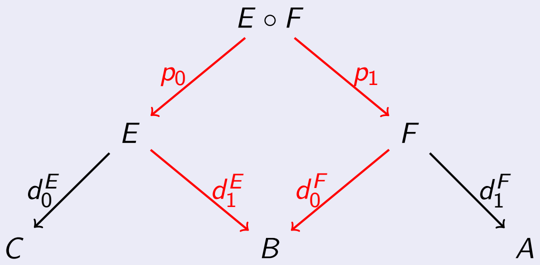
How to compose two relations:

tabulate the relations. . .



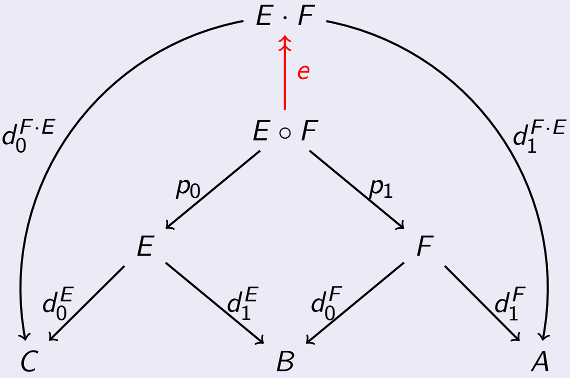
How to compose two relations:

. . . form the pullback. . .

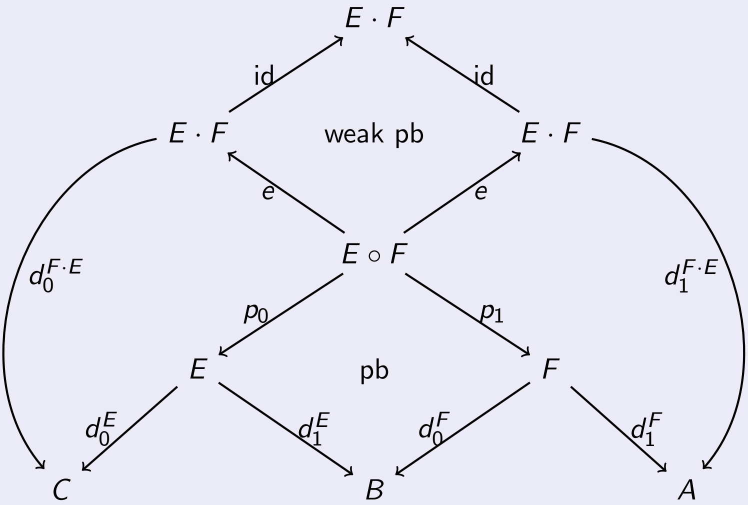


How to compose two relations:

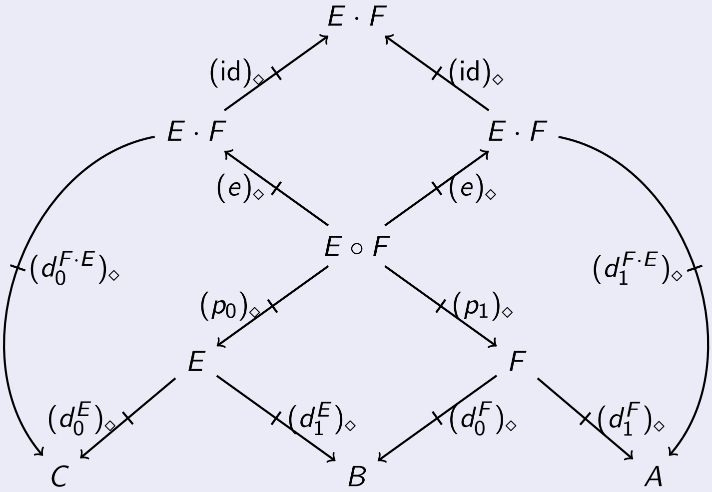
. . . form the quotient. . .



The composition diagram written more carefully



The presence of (weak) pullbacks in Set makes the following commutative in Rel(Set)



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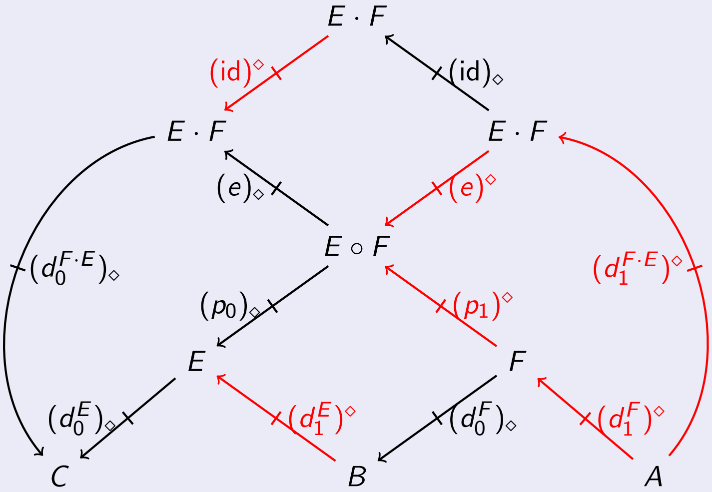

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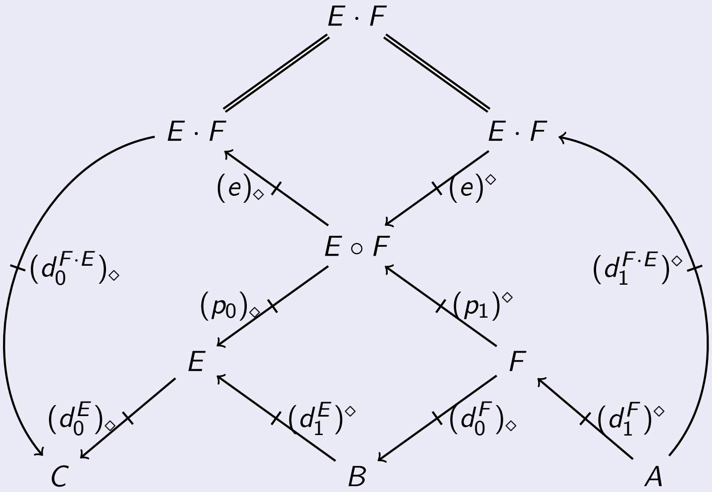

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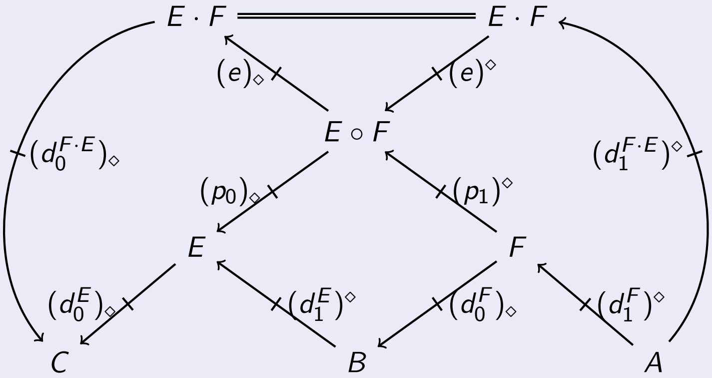
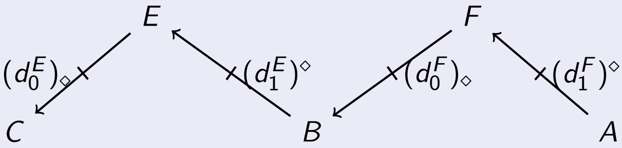


The previous makes composition work smoothly



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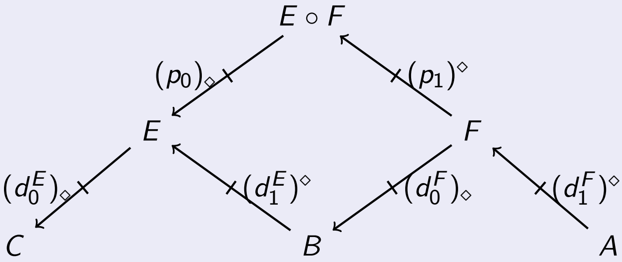

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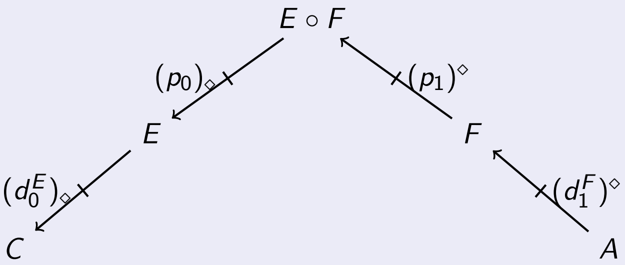

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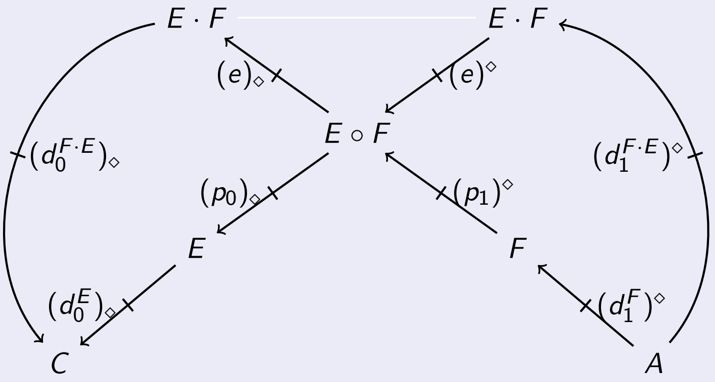

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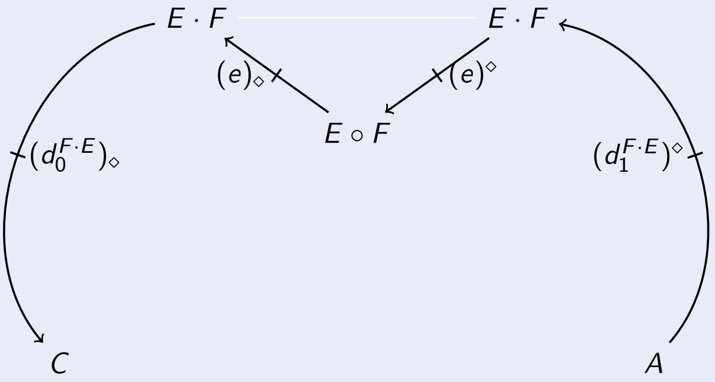

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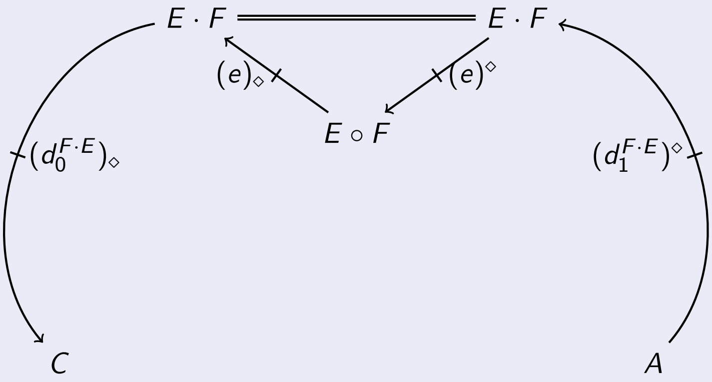

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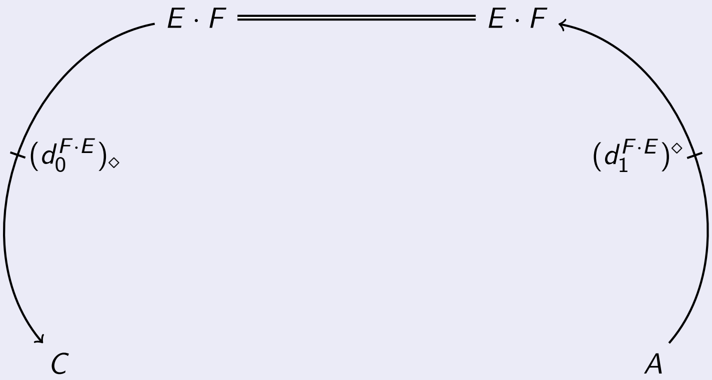
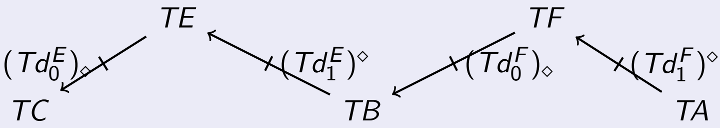


And if T preserves weak pullbacks, $\overline{T}$ preserves composition



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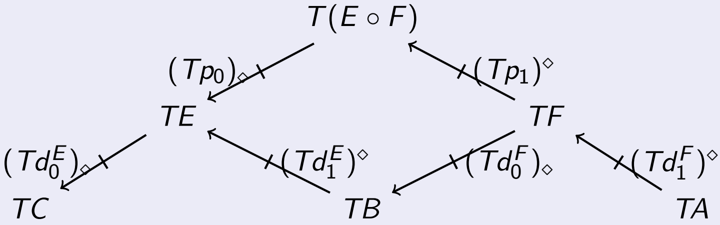

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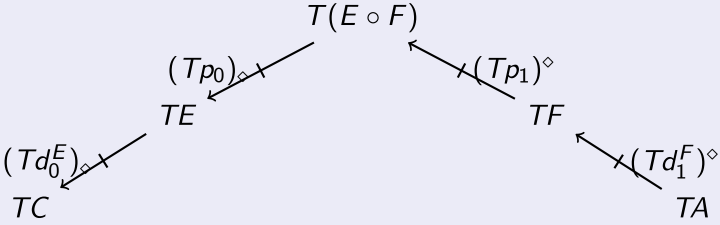

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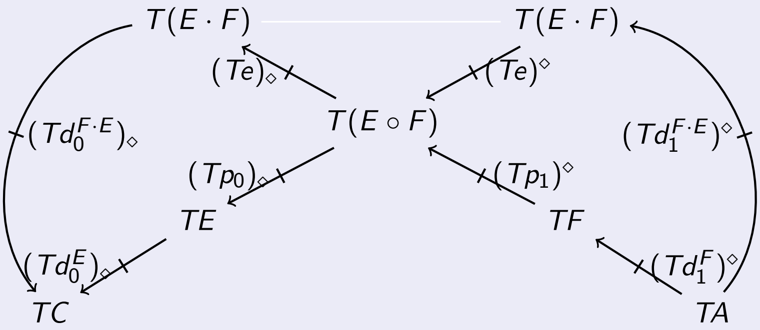

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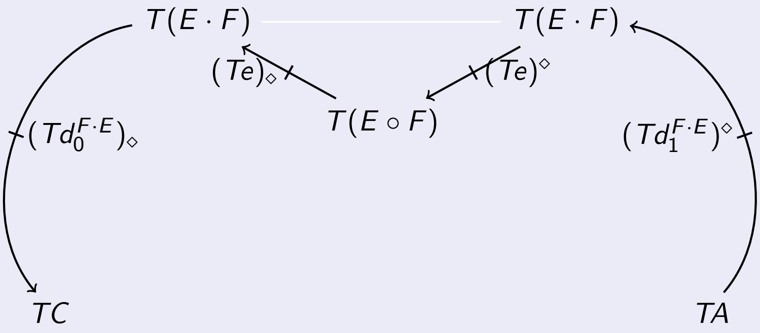

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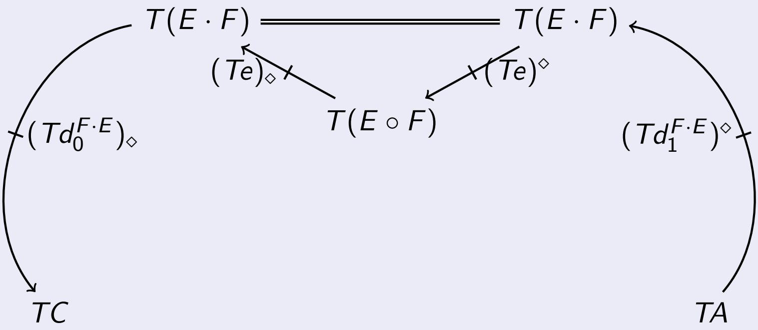

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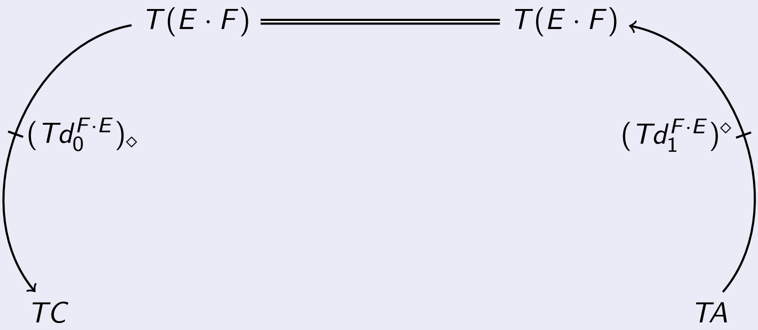


We want to pass from Set to more general categories to obtain more general applications.

The level of generality:

> Set is replaced by $\mathcal{V}$-cat, $\mathcal{V}$ being rather simple.

Problem:

> "Relations" can no longer be tabulated by spans, we need to cotabulate them by cospans.

Advantages:

1. Hermida's idea goes through with only small modifications.
2. All "Kripke-polynomial" functors on $\mathcal{V}$-cat admit a functorial relation lifting.

Definition

A commutative quantale[a] $\mathcal{V}$ is a tuple $(\mathcal{V}_o, \otimes, I, [-,-])$ where

1. $\mathcal{V}_o$ is a complete lattice.
2. The tensor $\otimes$ is commutative, associative, has I as a unit.
3. There is an adjunction $- \otimes a \dashv [a, -] : \mathcal{V}_o \longrightarrow \mathcal{V}_o$, i.e., $x \otimes a \leq y$ iff $x \leq [a, y]$ holds, for every a, x and y.

[a] Or, a commutative complete residuated lattice.

Examples

1. $\mathcal{V}_o$ = two-element chain, $\otimes$ = meet, I = top.
2. $\mathcal{V}_o$ = unit interval with reversed order, $\otimes$ = max, I = zero.
3. ... many others.

Definition

A small $\mathcal{V}$-category $\mathcal{A}$ consists of a small set of objects, a, b, ..., and $\mathcal{A}(a,b)$ in $\mathcal{V}_o$, for every pair a, b of objects, such that

1. $I \leq \mathcal{A}(a,a)$, for every a.
2. $\mathcal{A}(a,b) \otimes \mathcal{A}(b,c) \leq \mathcal{A}(a,c)$, for every a, b, c.

A $\mathcal{V}$-functor $f : \mathcal{A} \longrightarrow \mathcal{B}$ consists of an object-assignment $a \mapsto fa$ such that $\mathcal{A}(a,b) \leq \mathcal{B}(fa,fb)$ holds, for every a, b.

Small $\mathcal{V}$-categories and $\mathcal{V}$-functors form a 2-category

$$\mathcal{V}\text{-cat}$$

The 2-cell $f \to g$ witnesses the inequality $I \leq \bigwedge_x \mathcal{B}(fx, gx)$.

Examples

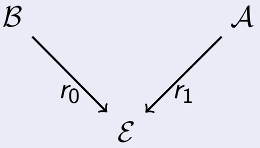
1. $\mathcal{V}_o$ = two-element chain, $\otimes$ = meet, I = top. Then $\mathcal{V}$-cat = preorders and monotone maps.
2. $\mathcal{V}_o$ = unit interval with reversed order, $\otimes$ = max, I = zero. Then $\mathcal{V}$-cat = ultrametric spaces and nonexpanding maps.
3. ... many others.

Definition

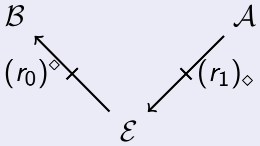
A relation[a] from $\mathcal{A}$ to $\mathcal{B}$ is a $\mathcal{V}$-functor $R : \mathcal{B}^{op} \otimes \mathcal{A} \longrightarrow \mathcal{V}$,

denoted by $R : \mathcal{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{B}$

Relation R is cotabulated by the cospan



if $R =$



where $(r_1)_\diamond(e, a) = \mathcal{E}(e, r_1(a))$, $(r_0)^\diamond(b, e) = \mathcal{E}(r_0(b), e)$.

[a] Or, module, or, profunctor, or, distributor.

Street's characterisation of relations in $\mathcal{V}$-cat (1980)

Relations in $\mathcal{V}$-cat correspond to cospans that are codiscrete cofibrations in $\mathcal{V}$-cat.
Composition of these cospans involves pushouts in $\mathcal{V}$-cat and fully-faithful $\mathcal{V}$-functors.
$\mathcal{V}$-functor $f : \mathcal{A} \longrightarrow \mathcal{B}$:

$$\mathcal{A}(a, b) \leq \mathcal{B}(fa, fb)$$

Fully-faithful $\mathcal{V}$-functor $f : \mathcal{A} \longrightarrow \mathcal{B}$:

$$\mathcal{A}(a, b) = \mathcal{B}(fa, fb)$$

(Weak) pullbacks are replaced by exact squares

A lax square
$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{p_1} & \mathcal{B} \\ {\scriptstyle p_0}\downarrow & \nearrow & \downarrow{\scriptstyle g} \\ \mathcal{A} & \xrightarrow[f]{} & \mathcal{C} \end{array}$$
in $\mathcal{V}$-cat is exact

iff the square
$$\begin{array}{ccc} \mathcal{P} & \overset{(p_1)^\diamond}{\longleftarrow\!\!\!\!\!\!+\;\;} & \mathcal{B} \\ {\scriptstyle (p_0)_\diamond}\downarrow & & \downarrow{\scriptstyle (g)_\diamond} \\ \mathcal{A} & \underset{(f)^\diamond}{\longleftarrow\!\!\!\!\!\!+\;\;} & \mathcal{C} \end{array}$$
commutes in Rel($\mathcal{V}$-cat)

iff, for all a and b
$$\mathcal{C}(fa, gb) = \bigvee_w \mathcal{A}(a, p_0(w)) \otimes \mathcal{B}(p_1(w), b).$$

The characterisation theorem

For a 2-functor $T : \mathcal{V}\text{-cat} \longrightarrow \mathcal{V}\text{-cat}$, the following are equivalent:

1. There is a 2-functor $\overline{T} : \mathsf{Rel}(\mathcal{V}\text{-cat}) \longrightarrow \mathsf{Rel}(\mathcal{V}\text{-cat})$ such that the square

$$\begin{array}{ccc} \mathsf{Rel}(\mathcal{V}\text{-cat}) & \overset{\overline{T}}{\dashrightarrow} & \mathsf{Rel}(\mathcal{V}\text{-cat}) \\ {\scriptstyle (-)_\diamond}\uparrow & & \uparrow{\scriptstyle (-)_\diamond} \\ \mathcal{V}\text{-cat} & \underset{T}{\longrightarrow} & \mathcal{V}\text{-cat} \end{array}$$

   commutes.

2. T preserves exact squares.

Here, for $f : \mathcal{A} \longrightarrow \mathcal{B}$, $f_\diamond(b, a) = \mathcal{B}(b, fa)$.

Definition of $\overline{T}$

Suppose $R : \mathcal{A} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathcal{B}$ is cotabulated by

$$\mathcal{B} \xrightarrow{r_0} \mathcal{E} \xleftarrow{r_1} \mathcal{A}$$

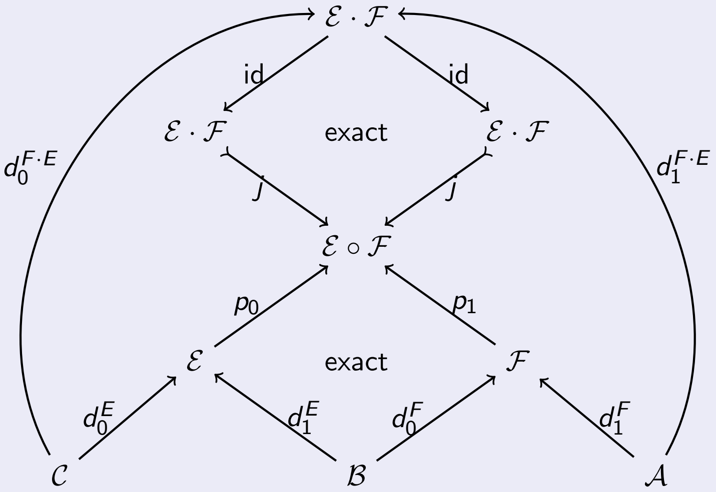
Define $\overline{T}(R) : T\mathcal{A} \longrightarrow\!\!\!\!\!\!\!+\;\; T\mathcal{B}$

as the composite

$$T\mathcal{B} \xleftarrow{(Tr_0)^\diamond} T\mathcal{E} \xleftarrow{(Tr_1)_\diamond} T\mathcal{A}$$

$$\begin{aligned} \overline{T}(R)(\beta, \alpha) &= \bigvee_w T\mathcal{E}(w, Tr_1(\alpha)) \otimes T\mathcal{E}(Tr_0(\beta), w) \\ &= T\mathcal{E}(Tr_0(\beta), Tr_1(\alpha)) \end{aligned}$$

The composition diagram



And the rest of the reasoning is analogous to sets.

Kripke-polynomial functors

All 2-functors $T : \mathcal{V}\text{-cat} \longrightarrow \mathcal{V}\text{-cat}$, given by

$$T ::= \mathsf{Id} \mid \mathsf{const}_{\mathcal{X}} \mid T + T \mid T \times T \mid T \otimes T \mid T^{\partial} \mid \mathcal{X} \mapsto [\mathcal{X}^{op}, \mathcal{V}]$$

where $T^{\partial}\mathcal{X} = (T(\mathcal{X}^{op}))^{op}$, preserve exact squares. Hence they give rise to a "well-behaved" coalgebraic cover modality.

Examples for preorders

1. All the Kripke-polynomial functors preserve exact squares.
2. The lowerset functor $\mathcal{L}\mathcal{X} = [\mathcal{X}^{op}, 2]$:

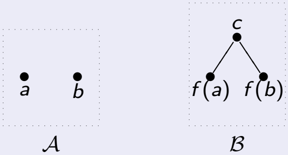
$$\overline{\mathcal{L}}(R)(B, A) \text{ iff } \forall b \in B\ \exists a \in A\ R(b, a)$$

3. The convex-set functor:

$$\overline{\mathcal{P}}(B, A) \text{ iff } \forall b \in B\ \exists a \in A\ R(b, a)\ \&\ \forall a \in A\ \exists b \in B\ R(b, a)$$

A counterexample for preorders

The connected-component functor does not preserve exact squares, since it does not preserve order embeddings, e.g., the embedding $f: \mathcal{A} \to \mathcal{B}$



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