Distributive-Law Semantics for Cellular Automata and Agent-Based Models

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Ecological Modelling
Universität Bayreuth

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Research Group

Come Again – Ecology??

- What can bring a **computer scientist** (compiler construction, functional programming) and an **ecologist** (forestry, soil science) together?
- Ecology has no theoretical background (or mathematicians) of its own.
- Theoretical concepts are supplied by the highest bidder.
- Current monopolist: **classical physics**.

  An ecosystem is physical, and accidentally alive.
- Hopeful contender: **computer science**.

  An ecosystem is an operating system on an earthly platform.

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Ecosystem modelling has **complementary** requirements:

State-based (physics) flows, laws, dynamics, prediction

Behaviour-based (CS) resources, actors, strategies, evaluation

Steps Taken

1. Map state & behaviour to initial algebra & final coalgebra, resp., for **pure** cases with running example (Hauhs and Trancón y Widemann 2010)
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Some Philosophy of Science

Another Distinction

- In sceptical science, two kinds of state should be distinguished:

  Ontic how things are; **cause** of behaviour
  Epistemic how things appear; **reflection** of behaviour

- Analogies to algebra–coalgebra distinction.

Danger

Arguments that fail to distinguish are vulnerable to *begging the question*:

- A person is (called) forgetful **because** he forgets things;
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- A veritable industry in social and environmental sciences
- Relationship to empirical approaches strained
  - great tool for demonstration of ideas
  - hardly any analytic/predictive value
- No commonly accepted definition

  pragmatic software done with agent techniques/tools/frameworks

  technical spatial OOP

  stylistic first-person narrative of cellular automata
- Program variables double as ontic and epistemic state!

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Breeding Synchrony (Jovani and Grimm 2008)

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- Birds live in a toroidal colony
- Individual behaviour is controlled by *stress level*
  - stress decreases as summer draws near
  - relaxed birds lay eggs
- Collective behaviour arises from stress distribution
  - stress is randomly distributed initially
  - stressed birds stress their neighbours
  - synchronous breeding emerges
- Is stress level ontic or epistemic?

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Observation

- ABMs frequently confuse kinds of state,
- but inconsistencies are hard to demonstrate!

Hypothesis 2

- The underlying theoretical structure of ABMs ensures consistency by construction.
- Its axioms need to be measured against the standards of the Scientific Method:

  bad unlikely to hold in reality
  worse impossible to test in reality

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Cellular Automata

- Identical Moore automata distributed in discrete space
  - dual views as local automata or global automaton
- Every cell has finitely many neighbours
  - many topologies studied (Tyler 2005)
  - current state of neighbours is input
- Spatial as well as temporal dynamics
  - initial distribution of states
  - mobile patterns

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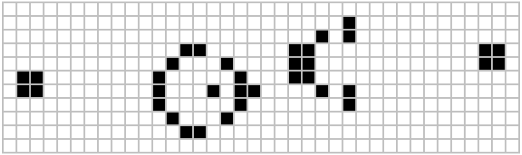

(Wikipedia 2011)

Bialgebraic Semantics in a Nutshell

Ingredients

1. A *syntax* functor Σ
2. A *behaviour* functor B
3. A distributive law $\lambda : \Sigma B \Rightarrow B\Sigma$

λ-Bialgebras

$$
\begin{array}{ccccc}
\Sigma X & \xrightarrow{f} & X & \xrightarrow{g} & BX \\
\downarrow{\scriptstyle \Sigma g} & & & & \uparrow{\scriptstyle Bf} \\
\Sigma BX & & \xrightarrow[\lambda_X]{} & & B\Sigma X
\end{array}
$$

- Σ-algebra f and B-coalgebra g commute, mediated by λ.
- g is a Σ-algebra morphism from f to $B^\lambda f = Bf \circ \lambda_X$.
- f is a B-coalgebra morphism to g from $\Sigma_\lambda g = \lambda_X \circ \Sigma g$.

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- Initial Σ-algebras extend uniquely to initial λ-bialgebras.
- Final B-coalgebras extend uniquely to final λ-bialgebras.
- There is a unique end-to-end λ-bialgebra homomorphism.

Research Agenda Revisited

Hypothesis 2.1

Distributive laws – the secret ingredient of ABMs?

Tasks

1. Give "natural" bialgebra semantics for CAs (here)
2. How does existence of λ perform as empirical axiom? (??)

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Local Research Agenda

Recipe for CA Semantics

1. Choose a functor Σ for the spatial arrangement of distributed state, over a local state set
2. Choose a functor B for the temporal behaviour of automata
3. Give distributive-law rules for the spatial language
4. Give a distributive-law rule for local transitions
5. Put everything together and obtain a unique homomorphism
6. Feed with global state & sequence of global inputs to obtain sequence of global states
   (initial & boundary conditions $\to$ trajectory)

Here proof-of-concept example for steps 1, 2, 3

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Syntax

- 2D regular grid with torsion

$$S ::= [L] \mid S \,|\, S \mid S \,/\, S \mid S^{\leftrightarrow} \mid S^{\updownarrow}$$

  - chosen as minimal non-trivial example
  - every term has well-defined *width*, *height*, and array-like element *selection*
  - avoid mismatched composition by padding with default $* \in L$
  - other types of torsion possible: *Möbius*, *solenoid*
- Fully compositional (unlike traditional frameworks)

$$A^{\leftrightarrow} \,/\, \left(B^{\leftrightarrow} \,|\, C^{\leftrightarrow}\right)$$

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Semantics

- Cellular automata are Moore-type (delayed I/O)

$$B \quad X = O \times X^{I}$$

- They consume observable neighbourhood state and produce observable own state
- Unified perspectives:

  local $S = \quad L$; neighbourhood = cells nearby

  global $S = WL$; neighbourhood = world boundaries

- Open questions:
  1. What is the neighbourhood functor C?
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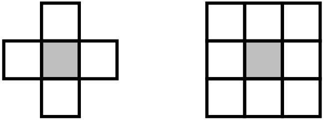

Moore
$CX = X^4$

von Neumann
$CX = X^8$

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$$\gamma : C^\sharp W \Rightarrow W C^\sharp \qquad \text{where } C^\sharp X = CX \times X$$

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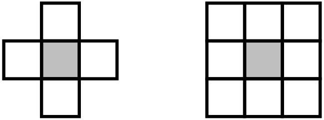

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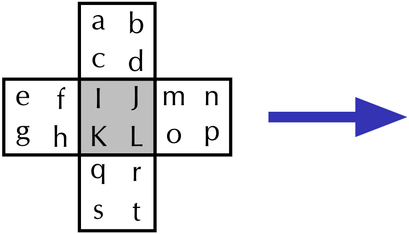

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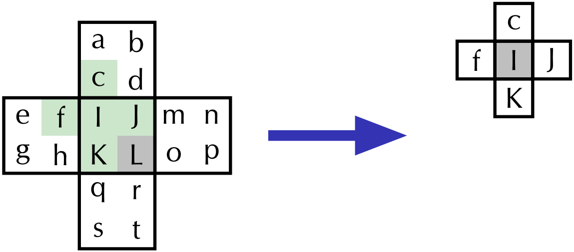

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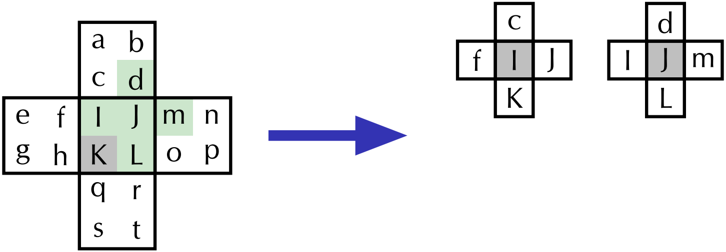

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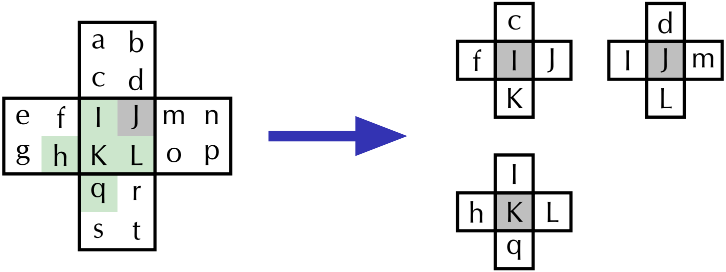

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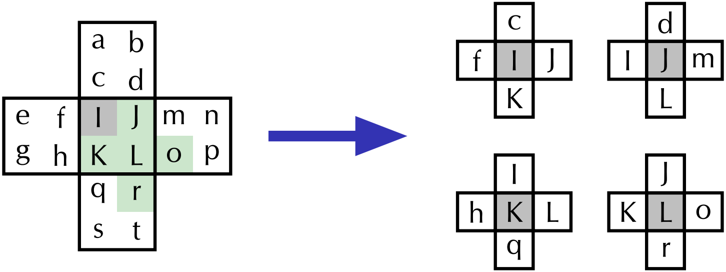

Neighbourhood & Transition

Automata Poetry

1. State transitions are algebraic
$$u : C^\sharp L \to L$$
2. Observability is coalgebraic
$$u^\rhd : L \to B_L^C L \qquad u^\rhd(x) = \big(x, u(\_, x)\big)$$
3. Globalization is (γ-)bialgebraic
$$W^\gamma u : C^\sharp W L \to W L$$

Example: Conway's Game of Life

$$L = \{0, 1\} \qquad u\big((a_1, \ldots, a_8), b\big) = \begin{cases} 1 & \sum a_i = 3 \\ b & \sum a_1 = 2 \\ 0 & \text{otherwise} \end{cases}$$

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Relative Addresses

Relative Addressing Theorem

Given the following:

- a *chart* $\chi : C\mathbb{Z}^2$ of relative coordinates,
- an *extended selection* $\mathsf{sl}^+ : C^\sharp WL \to \mathbb{Z}^2 \to L$,

one can define

- a natural transformation $\widehat{\chi}_L : WL \to WC^\sharp\mathbb{Z}^2$, inductively in Σ,
- a distributive law $\gamma : C^\sharp W \Rightarrow WC^\sharp$, namely

$$\gamma_L(c, x) = WC^\sharp\big(\mathsf{sl}^+(c, x)\big)\big(\widehat{\chi}_L(x)\big)$$

that satisfy the shapeliness conditions.

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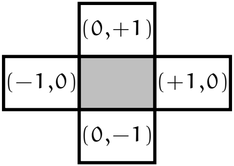

Compositionality

- For the desired spatio-temporal distributive law we need to lift syntax over globalized updates (co-syntax).
- Find a collection of natural transformations

$$\begin{aligned} \text{cosingleton} : &\quad CW \Rightarrow C \\ \text{cohwrap}, \text{covwrap} : &\quad W \times CW \Rightarrow CW \\ \text{cobeside}, \text{coabove} : &\quad W \times W \times CW \Rightarrow CW \end{aligned}$$

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$$\begin{aligned}\text{cosingleton}: &\quad CW \Rightarrow C\\ \text{cohwrap}, \text{covwrap}: &\quad W \times CW \Rightarrow CW\\ \text{cobeside}, \text{coabove}: &\quad W \times W \times CW \Rightarrow CW\end{aligned}$$

- such that, for globalized transitions $g = W^{\gamma} u$,

$$\begin{aligned}\left[u(\text{cosingleton}_L(c), a)\right] &= g\big(c, [a]\big)\\ g\big(\text{cohwrap}_L(x, c), x\big)^{\leftrightarrow} &= g(c, x^{\leftrightarrow})\\ g(c_1, x_1) \mid g(c_2, x_2) &= g(c, x_1 \mid x_2)\\ \text{where} \quad \text{cobeside}_L(x_1, x_2, c) &= (c_1, c_2)\end{aligned}$$

- This is easier than it looks!

Compositionality

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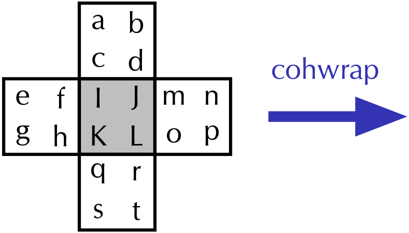
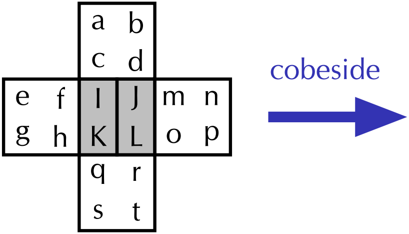
Illustration





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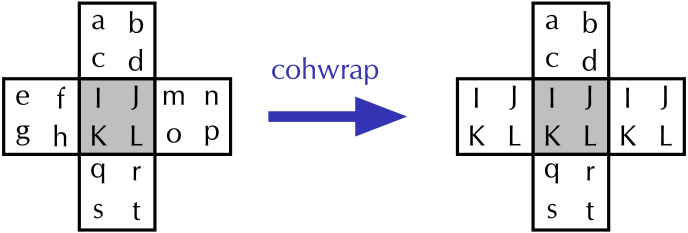

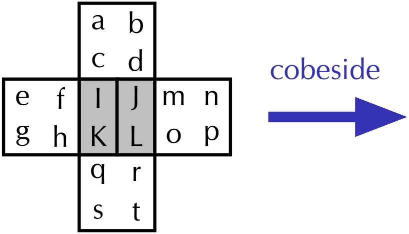

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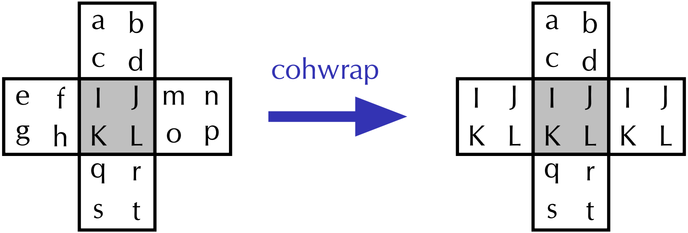

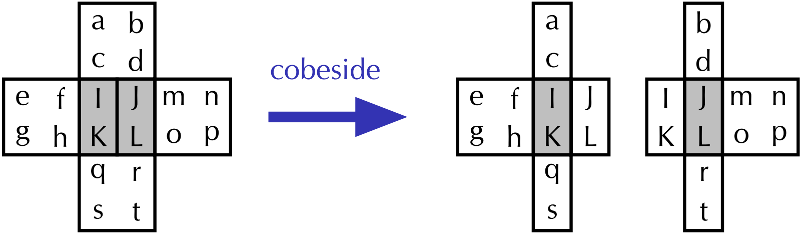

Distributive Law

$$\lambda^u : \Sigma_L B^C_{WL} \Rightarrow B^C_{WL} \Sigma_L$$

$$\frac{x_1 \xrightarrow{s_1} y_1 \quad x_2 \xrightarrow{s_2} y_2}{x_1 \mid x_2 \xrightarrow[\text{cobeside}]{s_1 \mid s_2} y_1 \mid y_2} \qquad \frac{x \xrightarrow{s} y}{x^{\leftrightarrow} \xrightarrow[\text{cohwrap}]{s^{\leftrightarrow}} y^{\leftrightarrow}}$$

$$\frac{}{[a] \xrightarrow[\text{cosingleton}]{[a]} [u(\_, a)]}$$

$$\frac{x_1 \xrightarrow{s_1} y_1 \quad x_2 \xrightarrow{s_2} y_2}{x_1 \mathbin{/} x_2 \xrightarrow[\text{coabove}]{s_1 / s_2} y_1 \mathbin{/} y_2} \qquad \frac{x \xrightarrow{s} y}{x^{\updownarrow} \xrightarrow[\text{cohwrap}]{s^{\updownarrow}} y^{\updownarrow}}$$

Comments

- Formal definition of rule format
- Local transition relevant to singleton case only
- World shape is observed and preserved
- Post-states are mediated by co-syntax

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Proof of Equivalence

Classical Specification

$$u : C^{\sharp}L \to L$$

Distributive Specification

$$h^{u} : \mu\Sigma_{L} \to \nu B^{C}_{WL}$$

Equivalence Theorem

$$j^{u} = h^{u}$$

- Proof Idea: $(W^{\curlyvee}u)^{\triangleright}$ is the coalgebra part of the initial λ-bialgebra $\Longrightarrow$ induction $\Longrightarrow$ coinduction.
- Amounts to showing that rules for λ^{u} and co-syntax cancel out.

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Summary

- High-level specification of CA semantics in terms of distributive-laws

  topological (γ) neighbourhood over world

  dynamical (λ) space over time

- Correspond to basic evaluation algorithms
  - array loops with index manipulation
  - divide & conquer
- Equivalence
  - proof strictly follows bialgebraic structure
- Basic categorical bialgebra
  - can be implemented directly in Haskell
  - first real instance of bialgebraic EDSL? (Jaskelioff, Ghani, and Hutton 2011)
  - watch out for forthcoming paper!

Conclusion

Suggested Extensions to CA Theory

- Weird topological operators
  - add clauses to Σ
- Unobservable state
  - insert projection into output of $\_^{\triangleright}$
- Dynamic shape & topology
  - drop shape-preservation of λ

Open Philosophical Question

- ABMs do in fact have a consistent mapping between ontic and epistemic states, but
- when axiomatically assuming the existence of a spatio-temporal distributive law, what are we saying about the world?

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Beware of foul models!

Questions?

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